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FACTORING MULTIVARIATE INTEGRAL POLYNOMIALS, II

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# Factoring multivariate integral polynomials, II <sup>\*)</sup>

by

A.K. Lenstra

## ABSTRACT

We show that the problem of factoring multivariate integral polynomials can be reduced in polynomial-time to the univariate case. Our reduction makes use of lattice techniques as introduced in [3].

KEY WORDS & PHRASES: *polynomial algorithm, polynomial factorization*

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<sup>\*)</sup> This report will be submitted for publication elsewhere.

## 1. Introduction.

In [5] we presented a polynomial-time algorithm to factor polynomials in  $\mathbb{Z}[X, Y]$ , and we pointed out how to generalize the algorithm to  $\mathbb{Z}[X_1, X_2, \dots, X_t]$  for  $t \geq 3$ . A nice feature of this algorithm is that it doesn't depend on the polynomial-time algorithm to factor in  $\mathbb{Z}[X]$  (cf. [3]).

Instead of working out the details of this direct approach for  $t \geq 3$  (this will be done for  $\mathbb{Q}(\alpha)[X_1, X_2, \dots, X_t]$  in a forthcoming paper [6]), we here simplify the method from [5] somewhat, which results in a polynomial-time reduction from factoring in  $\mathbb{Z}[X_1, X_2, \dots, X_t]$  to factoring in  $\mathbb{Z}[X]$ . This reduction is similar to the reduction from  $\mathbb{F}_q[X_1, X_2, \dots, X_t]$  to  $\mathbb{F}_q[X, Y]$  that was given in [4].

An outline of our reduction is as follows. First we evaluate the polynomial  $f \in \mathbb{Z}[X_1, X_2, \dots, X_t]$  in a suitably chosen integer point  $(X_2 = s_2, X_3 = s_3, \dots, X_t = s_t)$ , to obtain a polynomial  $\tilde{f} \in \mathbb{Z}[X_1]$ . Using the algorithm from [3] we then compute an irreducible factor  $\tilde{h} \in \mathbb{Z}[X_1]$  of  $\tilde{f}$ . Next we construct an integral lattice containing the factor  $h_0$  of  $f$  that corresponds to  $\tilde{h}$ , and we prove that  $h_0$  is the shortest vector in this lattice. As usual, this enables us to compute  $h_0$  by means of the so-called *basis reduction algorithm* (cf. [3: Section 1]; in the sequel we will assume the reader to be familiar with this basis reduction algorithm and its properties).

## 2. Factoring multivariate integral polynomials.

Let  $f \in \mathbb{Z}[X_1, X_2, \dots, X_t]$  be the polynomial to be factored, with the number of variables  $t \geq 2$ . By  $\delta_i f = n_i$  we denote the degree of  $f$  in  $X_i$ . We

often use  $n$  instead of  $n_1$ . We put  $N_i = \prod_{k=i}^t (n_k + 1)$ , and  $N = N_1$ . The content  $\text{cont}(f) \in \mathbb{Z}[X_2, X_3, \dots, X_t]$  of  $f$  is defined as the greatest common divisor of the coefficients of  $f$  with respect to  $X_1$ ; we say that  $f$  is *primitive* if  $\text{cont}(f) = 1$ .

Without loss of generality we may assume that  $2 \leq n_i \leq n_{i+1}$  for  $1 \leq i < t$ , and that the gcd of the integer coefficients of  $f$  equals one.

We present an algorithm to factor  $f$  into its irreducible factors in  $\mathbb{Z}[X_1, X_2, \dots, X_t]$  that is polynomial-time in  $N$  and the size of the integer coefficients of  $f$ .

Let  $s_2, s_3, \dots, s_t \in \mathbb{Z}_{>0}$  be a  $(t-1)$ -tuple of integers. For  $g \in \mathbb{Z}[X_1, X_2, \dots, X_t]$  we denote by  $\tilde{g}_j$  the polynomial  $g$  modulo  $((X_2 - s_2), (X_3 - s_3), \dots, (X_j - s_j)) \in \mathbb{Z}[X_1, X_{j+1}, X_{j+2}, \dots, X_t]$ ; i.e.  $\tilde{g}_j$  is  $g$  with  $s_i$  substituted for  $X_i$  for  $i = 2, 3, \dots, j$ . Notice that  $\tilde{g}_1 = g$ , and that  $\tilde{g}_j = \tilde{g}_{j-1}$  modulo  $(X_j - s_j)$ . We put  $\tilde{g} = \tilde{g}_t$ .

Suppose that an irreducible, primitive factor  $\tilde{h} \in \mathbb{Z}[X_1]$  of  $f$  is given such that

$$(2.1) \quad \tilde{h}^2 \text{ doesn't divide } f \text{ in } \mathbb{Z}[X_1], \text{ and } \delta_1 \tilde{h} > 0.$$

This condition implies that there exists an irreducible factor  $h_0 \in \mathbb{Z}[X_1, X_2, \dots, X_t]$  of  $f$  such that  $\tilde{h}$  divides  $h_0$  in  $\mathbb{Z}[X_1]$ , and that this polynomial  $h_0$  is unique up to sign.

(2.2) Let  $m$  be an integer with  $\delta_1 \tilde{h} \leq m < n$ . We define  $L$  as the collection of polynomials  $g$  in  $\mathbb{Z}[X_1, X_2, \dots, X_t]$  such that

- (i)  $\delta_1 g \leq m$ , and  $\delta_i g \leq n_i$  for  $2 \leq i \leq t$ ,
- (ii)  $\tilde{h}$  divides  $\tilde{g}$  in  $\mathbb{Z}[X_1]$ .

This is a subset of the  $(m+1)N_2$ -dimensional real vector space  $\mathbb{R} + \mathbb{R}X_t + \dots +$

$\mathbb{R}X_1^m X_2^{n_2} \dots X_t^{n_t}$ . We put  $M = (m+1)N_2$ . This vector space can be identified with  $\mathbb{R}^M$  by identifying the polynomial  $\sum_{i=0}^m \sum_{j=0}^{n_2} \dots \sum_{k=0}^{n_t} a_{ij\dots k} X_1^i X_2^j \dots X_t^k \in \mathbb{R}[X_1, X_2, \dots, X_t]$  with the  $M$ -dimensional vector  $(a_{00\dots 0}, a_{00\dots 1}, \dots, a_{mn_2\dots n_t})$ . The collection  $L$  is a lattice in  $\mathbb{Z}^M$  of rank  $M - \delta_1 n$ , and a basis for  $L$  is given by

$$\{X_1^i \prod_{j=2}^t (X_j - s_j)^{i_j} : 0 \leq i \leq m, 0 \leq i_j \leq n_j \text{ for } 2 \leq j \leq t, \text{ and}$$

$$(i_2, i_3, \dots, i_t) \neq (0, 0, \dots, 0)\}$$

$$\cup \{n X_1^{i-\delta_1 n} : \delta_1 n \leq i \leq m\}$$

(cf. [4: (3.2)]).

We define the *length*  $|g|$  of the vector associated with the polynomial  $g$  as the ordinary Euclidean length of this vector. The *height*  $g_{\max}$  is defined as the largest absolute value of any of the integer coefficients of  $g$ .

(2.3) Proposition. Suppose that  $b$  is a non-zero element of  $L$  such that

$$(2.4) \quad s_j \geq f_{\max}^m b_{\max}^n (n+m)! (N_2 \prod_{i=2}^{j-1} s_i^{n_i})^{n+m}$$

for  $2 \leq j \leq t$ . Then  $\gcd(f, b) \neq 1$  in  $\mathbb{Z}[X_1, X_2, \dots, X_t]$ . <sup>\*</sup>

Proof. Suppose on the contrary that  $\gcd(f, b) = 1$ . This implies that the resultant  $R = R(f, b) \in \mathbb{Z}[X_2, X_3, \dots, X_t]$  of  $f$  and  $b$  (with respect to the variable  $X_1$ ) is unequal to zero.

We derive an upper bound for  $(\tilde{R}_j)_{\max}$ . Because  $\tilde{R}_j$  is the resultant of  $\tilde{f}_j$  and  $\tilde{b}_j$  we have

$$(2.5) \quad (\tilde{R}_j)_{\max} \leq (\tilde{f}_j)_{\max}^m (\tilde{b}_j)_{\max}^n (n+m)! N_{j+1}^{n+m-2}$$

<sup>\*</sup>) Here, and in the sequel,  $f_{\max}^m$  denotes  $(f_{\max})^m$ .

as is easily verified. Because  $\tilde{b}_j = \tilde{b}_{j-1} \text{ modulo } (X_j - s_j)$ , we have

$$(\tilde{b}_j)_{\max} \leq (\tilde{b}_{j-1})_{\max} (n_j + 1) s_j^{n_j},$$

so that

$$(2.6) \quad (\tilde{b}_j)_{\max} \leq b_{\max} \prod_{i=2}^j (n_i + 1) s_i^{n_i},$$

and similarly

$$(2.7) \quad (\tilde{f}_j)_{\max} \leq f_{\max} \prod_{i=2}^j (n_i + 1) s_i^{n_i}.$$

Combining (2.5), (2.6), and (2.7), we obtain

$$(2.8) \quad (\tilde{R}_j)_{\max} < f_{\max}^m b_{\max}^n (n+m)! (N_2 \prod_{i=2}^j s_i^{n_i})^{n+m},$$

for  $1 \leq j < t$ .

Because  $\tilde{n}$  divides both  $\tilde{f}$  and  $\tilde{b}$  ((2.2)(ii)), we have that  $\tilde{R} = 0$ . But also  $R \neq 0$ , so there must be an index  $j$  with  $2 \leq j \leq t$  such that  $s_j$  is a zero of  $\tilde{R}_{j-1}$ . This implies that

$$|s_j| \leq (\tilde{R}_{j-1})_{\max}$$

for some  $j$  with  $2 \leq j \leq t$ , which yields, combined with (2.4) and (2.8), a contradiction. We conclude that  $\gcd(f, b) \neq 1$ .  $\square$

(2.9) Proposition. Let  $b_1, b_2, \dots, b_M$  be a reduced basis for  $L$  (cf. [3: Section 1]), where  $L$  and  $M$  are defined as in (2.2). Suppose that

$$(2.10) \quad s_j \geq f_{\max}^m ((M 2^{M-1})^{\frac{1}{2}} f_{\max})^n (n+m)! \left( e^{\sum_{i=1}^t n_i} N_2 \prod_{i=2}^{j-1} s_i^{n_i} \right)^{n+m}$$

for  $2 \leq j \leq t$ , and that  $f$  doesn't contain multiple factors. Then

$$(2.11) \quad (b_1)_{\max} \leq (M 2^{M-1})^{\frac{1}{2}} e^{\sum_{i=1}^t n_i} f_{\max}$$

and  $h_0$  divides  $b_1$ , if and only if  $\delta_1 h_0 \leq m$ .

Proof. If  $h_0$  divides  $b_1$ , then  $\delta_1 h_0 \leq \delta_1 b_1 \leq m$ ; this proves the "only if"-part.

We prove the "if"-part. Suppose that  $\delta_1 h_0 \leq m$ . The polynomial  $h_0$  is a divisor of  $f$ , so that

$$(h_0)_{\max} \leq e^{\sum_{i=1}^t n_i} f_{\max}$$

according to [2]. With  $\delta_1 h_0 \leq m$  and  $\delta_i h_i \leq n_i$  for  $2 \leq i \leq t$  we get

$$|h_0| \leq M^{\frac{1}{2}} e^{\sum_{i=1}^t n_i} f_{\max},$$

so that [3: (1.11)] combined with  $h_0 \in L$  (this follows from  $\delta_1 h_0 \leq m$ ) yields

$$|b_1| \leq (M 2^{M-1})^{\frac{1}{2}} e^{\sum_{i=1}^t n_i} f_{\max}.$$

This proves (2.11) because  $(b_1)_{\max} \leq |b_1|$ . With (2.10) and (2.3) we now have that  $\gcd(f, b_1) \neq 1$ . Suppose that  $h_0$  doesn't divide  $r = \gcd(f, b_1)$ . Then  $\tilde{r}$  divides  $f/\tilde{r}$ , so that, with

$$(f/r)_{\max} \leq e^{\sum_{i=1}^t n_i} f_{\max},$$

and (2.10), (2.11), and (2.3), we get that  $\gcd(f/r, b_1) \neq 1$ . This is a contradiction with  $r = \gcd(f, b_1)$ , because  $f$  doesn't contain multiple factors.  $\square$

(2.12) Suppose that  $f$  doesn't contain multiple factors and that  $f$  is primitive. Let  $s_2, s_3, \dots, s_t$  and  $\tilde{r}$  be chosen such that (2.10) with  $m$  replaced by  $n-1$  and (2.1) are satisfied. The divisor  $h_0$  of  $f$  can be



determined in the following way.

For the values  $m = \delta_1 \bar{n}, \delta_1 \bar{n} + 1, \dots, n-1$  in succession we apply the basis reduction algorithm (cf. [3: Section 1]) to the lattice  $L$  as defined in (2.2). We stop as soon as a vector  $b_1$  is found satisfying (2.11). It is not difficult to see that the first vector  $b_1$  satisfying (2.11) that we encounter, also satisfies  $b_1 = \pm h_0$  (here we apply [3: (1.37)] and (2.9)). If no vector satisfying (2.11) is found, then  $\delta_1 h_0 > n-1$ , so that  $h_0 = f$ ; this follows from (2.9).

(2.13) Proposition. Assume that the conditions in (2.12) are satisfied. The polynomial  $h_0$  can be computed in  $O((\delta_1 h_0 N_2)^4 \log B)$  arithmetic operations on integers having binary length  $O(N \log B)$ , where

$$\log B = O(\log f_{\max} + n + \log N_2 + \sum_{i=2}^t n_i \log s_i).$$

Proof. Combining

$$|\bar{n}| \leq \binom{2n}{n}^{\frac{1}{2}} |f|$$

(cf. [7]) and (2.7), we find that

$$|\bar{n}| \leq f_{\max} ((n+1) \binom{2n}{n})^{\frac{1}{2}} \prod_{i=2}^t (n_i+1) s_i^{n_i}.$$

The proof follows now immediately from (2.2), [3: (1.26)] and [3: (1.37)].  $\square$

(2.14) We describe an algorithm to compute the irreducible factors of  $f$  in  $\mathbb{Z}[X_1, X_2, \dots, X_t]$ . Assume that  $f$  is primitive.

First we compute the resultant  $R = R(f, f') \in \mathbb{Z}[X_2, X_3, \dots, X_t]$  of  $f$  and its derivative  $f'$  with respect to  $X_1$ , using the subresultant algorithm from [1]. We may assume that  $R \neq 0$ , i.e.  $f$  doesn't contain multiple

factors. (In the case that  $R=0$ , the greatest common divisor  $g$  of  $f$  and  $f'$  is also computed by the subresultant algorithm, and the factoring algorithm can be applied to  $f/g$ .)

Next we determine  $s_2, s_3, \dots, s_t \in \mathbb{Z}$  such that  $\tilde{R} \neq 0$  and such that (2.10) is satisfied with  $m$  replaced by  $n-1$ :

$$(2.15) \quad s_j \geq (nN_2 2^{nN_2-1})^{n/2} (2n-1)! (e^{\sum_{i=1}^t n_i} f_{\max} N_2 \prod_{i=2}^{j-1} s_i^{n_i})^{2n-1}$$

for  $2 \leq j \leq t$ . It follows from the reasoning in the proof of (2.3) that if we take  $s_j \in \mathbb{Z}_{>0}$  minimal such that (2.15) is satisfied, then  $\tilde{R} \neq 0$ .

By means of the algorithm from [3] we compute the irreducible and primitive factors of  $f$  of degree  $> 0$  in  $X_1$ . The condition  $\tilde{R} \neq 0$  implies that (2.1) holds for every irreducible factor  $\tilde{h}$  of  $\tilde{f}$  thus found.

Finally, the factorization of  $f$  is determined by repeated application of the algorithm described in (2.12).

(2.16) Theorem. Let  $f$  be a polynomial in  $\mathbb{Z}[X_1, X_2, \dots, X_t]$  with  $t \geq 2$ ,  $\delta_i f = n_i$ , and  $2 \leq n = n_1 \leq n_2 \leq \dots \leq n_t$ . The irreducible factorization of  $f$  can be found in  $O(n^{t-2} (N^6 + N^5 \log f_{\max}))$  arithmetic operations on integers having binary length  $O(n^{t-2} (N^3 + N^2 \log f_{\max}))$ , where  $N = \prod_{i=1}^t (n_i + 1)$ .

Remark. Because  $n^t = O(N)$ , Theorem (2.16) implies that  $f$  can be factored in time polynomial in  $N$  and  $\log f_{\max}$ .

Proof of (2.16). First assume that  $f$  is primitive. The resultant  $R$  can be computed in  $O(n^{3t-1} N_2^2)$  arithmetic operations on integers having binary length  $O(n^2 \log(f_{\max} N_2))$  (cf. [1]).

From the choice of  $s_j$  (cf. (2.15)) we derive

$$\log s_j = O(n^2 N_2 + n \log f_{\max} + \sum_{i=2}^{j-1} n n_i \log s_i)$$

for  $2 \leq j \leq t$ , so that

$$\log s_j = O((n^2 N_2 + n \log f_{\max}) \prod_{i=2}^{j-1} (1 + n n_i)).$$

This yields

$$(2.17) \quad \sum_{i=2}^t n_i \log s_i = O(n^{t-2} (N^2 + N \log f_{\max})),$$

which gives, combined with (2.7),

$$(2.18) \quad \log f_{\max} = O(n^{t-2} (N^2 + N \log f_{\max})).$$

The polynomial  $f$  can be factored in  $O(n^6 + n^5 \log f_{\max})$  arithmetic operations on integers having binary length  $O(n^3 + n^2 \log f_{\max})$ , according to [3: (3.6)].

With (2.18) this becomes

$$O(n^{t+3} (N^2 + N \log f_{\max}))$$

arithmetic operations on integers having binary length

$$O(n^t (N^2 + N \log f_{\max})).$$

According to (2.13) and (2.17), repeated application of the algorithm described in (2.12) takes

$$O(n^{t-2} (N^6 + N^5 \log f_{\max}))$$

arithmetic operations on integers having binary length

$$O(n^{t-2} (N^3 + N^2 \log f_{\max})).$$

The cost of applying (2.12) therefore dominates the costs of the computation of  $R$  and the factorization of  $f$ .

The same estimates are valid in the case that  $R=0$ . In this case we have that

$$(f/g)_{\max} \leq e^{\sum_{i=1}^t n_i} f_{\max}$$

(cf. [2]), so that the same estimates as above are valid for the computation of the factorization of  $f/g$ .

Finally, we consider the case that the content of  $f$  is unequal to one. The computation of  $\text{cont}(f)$  can be done in  $O(n n_2^{3t-4} N_3^2)$  arithmetic operations on integers having binary length  $O(n_2^2 \log(f_{\max} N_3))$  (cf. [1]). Because  $\delta_i f = \delta_i \text{cont}(f) + \delta_i (f/\text{cont}(f))$  for  $2 \leq i \leq t$ , the proof follows by repeated application of the above reasoning.  $\square$

(2.19) Remark. As mentioned in the introduction, a somewhat more complicated but similar approach leads to an algorithm that doesn't depend on the polynomial-time algorithm for factoring in  $\mathbb{Z}[X]$ . Instead, it can be seen as a direct generalization of the  $\mathbb{Z}[X]$ -algorithm. We won't give a detailed description of this alternative method here, we only indicate the main differences.

The divisor  $\tilde{h} \in \mathbb{Z}[X_1]$  of  $f$  is replaced by a divisor  $(\tilde{h} \bmod p^k) \in (\mathbb{Z}/p^k \mathbb{Z})[X_1]$  of  $(f \bmod p^k)$ , for some suitably chosen prime power  $p^k$ . Condition (2.2)(ii) is therefore replaced by the condition that  $(\tilde{h} \bmod p^k)$  divides  $(\tilde{g} \bmod p^k)$  in  $(\mathbb{Z}/p^k \mathbb{Z})[X_1]$ . The lattice  $L \subset \mathbb{Z}^M$  now has rank  $M$ , and a basis for  $L$  is given by

$$\{p^k x_1^i : 0 \leq i < \delta_1 \tilde{h}\}$$

$$\cup \{ (\tilde{h} \bmod p^k) x_1^{i-\delta_1 \tilde{h}} : \delta_1 \tilde{h} \leq i \leq m \}$$

$$\cup \{ x_1^i \prod_{j=2}^t (x_j - s_j)^{i_j} : 0 \leq i \leq m, 0 \leq i_j \leq n_j \text{ for } 2 \leq j \leq t, \text{ and}$$

$$(i_2, i_3, \dots, i_t) \neq (0, 0, \dots, 0) \}.$$

Again, it can be proven that, if  $s_2, s_3, \dots, s_t$  and  $p^k$  are sufficiently large, then the irreducible factor of  $f$  that corresponds to  $(\tilde{h} \bmod p^k)$  is the shortest vector in  $L$ . This factor can therefore be found by means of the basis reduction algorithm, and the resulting algorithm appears to be polynomial-time. For  $f \in \mathbb{Z}[X, Y]$  the details are given in [5], and for  $f \in \mathbb{Q}(\alpha)[X_1, X_2, \dots, X_t]$  in [6].

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